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Spin glass properties of a class of mean-field models

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Abstract. A class of mean-field models for spin glasses, in which the magnitude of the Onsager term is controlled to interpolate between the TAP equations for the SK model and the 'naive' mean-field (NMF) equations, is introduced. We examine the number of metastable states and the related parameters of the models. For the NMF model the spin glass susceptibility is also evaluated numerically. The results indicate that the rugged free energy structure and the marginal stability of the spin glass phase are common properties of these mean-field models, and that the Onsager term plays a minor role on yielding these novel spin glass properties.

1. Introduction

The mean-field theory based on the Sherrington–Kirkpatrick (SK) [1] model has revealed novel properties of the spin glass phase [2]. Among them are the rugged free energy structure with an infinite number of metastable states and the marginal stability of the phase. These properties are expected to be common to various randomly frustrated systems. But there are few models other than the SK model for which we have succeeded to formulate these properties.

In the TAP equations [3], which are the equations of state of the SK model written in terms of the mean site magnetisations $\{m_i\}$, there exists the Onsager term, which properly subtracts the reaction field from the ordinary mean (Weiss) field. Unfortunately, this term makes it rather hard for us to solve the equations explicitly [4, 5]. In fact it introduces, in configuration space, a large area in which the TAP equations themselves lose their validity. At the same time, however, this term has been thought to play a certain crucial role on the spin glass properties mentioned above, particularly on the marginal stability.

The equations of state without the Onsager term have been used to examine some spin glass properties since they are found to be numerically much more robust than the TAP equations [6, 7]. They are called the 'naive' mean-field (NMF) equations, and the properties of the equations themselves have been investigated in detail by Bray, Sompolinsky and Yu (BSY) [8]. They have introduced a spin model, for which the NMF equations hold true rigorously. This spin model, which we call the BSY model, is shown to have strikingly similar spin glass properties to those of the SK model.

In the present work we investigate spin glass properties of the NMF model in further detail. For this purpose it is helpful and instructive to introduce a class of mean-field models which interpolates the SK and NMF models. They are defined, without specifying any explicit spin models lying behind them, by the following equations of state and the corresponding free energy per site (divided by temperature with $k_B = 1$), f, written in terms of $\{m_i\}$ with $|m_i| \leq 1$:

$$G_i \equiv \tanh^{-1} m_i + \gamma\beta^2 J^2 (1-q) m_i - \beta \sum_j J_{ij} m_j \equiv g_\gamma(m_i) - \beta \sum_j J_{ij} m_j = 0 \tag{1.1}$$

and

$$f = \frac{1}{2N} \sum_i \{(1+m_i)\ln[\tfrac{1}{2}(1+m_i)] + (1-m_i)\ln[\tfrac{1}{2}(1-m_i)]\} - \frac{\beta}{N} \sum_{(ij)} J_{ij} m_i m_j - \tfrac{1}{4}\gamma\beta^2 J^2 (1-q^2) \tag{1.2}$$

where $\beta = 1/T$, $q = (1/N)\sum_i m_i^2$ and N is the number of sites in the system. The infinite-ranged interactions $\{J_{ij}\}$ are independent Gaussian random variables with the mean zero and the variance J^2/N. The external parameter γ in the above equations, which specifies a model, controls the magnitude of the Onsager term. With $\gamma = 1$ and 0 the equations reduce to the TAP and NMF equations, respectively.

The total number of metastable states averaged over the realisations of $\{J_{ij}\}$, $\langle N_s \rangle_J$, is examined following the method by Bray and Moore [9]. In the method $\langle N_s \rangle_J$ is given in terms of the order parameter q and other parameters, which are in turn determined by a set of stationary equations. We find several solutions, which include the SK, Sommers [10], and Bray and Moore [9] solutions. The largest $\langle N_s \rangle_J$ obtained is found to be quite insensitive to γ, i.e., its dependence on the reduced temperature is quite similar to that in the SK model [9]. We also investigate numerically the spin glass susceptibility of the NMF model in order to check the stability of its spin glass phase. It is also shown to be marginally stable, as is that of the SK model [4]. These results indicate that the novel spin glass properties mentioned above are common to the present class of mean-field models.

In the next section we examine the total number of metastable states in the models. Some comments on mathematical details are presented in the Appendix. In § 3 the stability of the NMF spin glass phase is analysed, and § 4 is devoted to concluding remarks.

2. Metastable states of the mean-field models

By means of the method of Bray and Moore (BM1) [9], the number of metastable states for a fixed value of the free energy f, $N_s(f)$, in the mean-field model introduced in §1 is given by

$$N_s(f) = N^2 \int_0^1 \mathrm{d}q \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}u}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \prod_i \left(\frac{\mathrm{d}x_i}{2\pi\mathrm{i}}\right) \int_{-1}^{1} \prod_i (\mathrm{d}m_i) \times \exp\Big(-N(\lambda q + uf) + \lambda \sum_i m_i^2 + u \sum_i f_\gamma(m_i) + \sum_i x_i g_\gamma(m_i) - \beta \sum_{(ij)} J_{ij}(x_i m_j + x_j m_i)\Big) \det \mathbf{A}\{J_{ij}\} \tag{2.1}$$

$$f_\gamma(m_i) = -\ln 2 + \tfrac{1}{2}\ln(1-m_i^2) + \tfrac{1}{2} m_i \tanh^{-1} m_i - \tfrac{1}{4}\gamma\beta^2 J^2 (1-q^2) \tag{2.2}$$

where **A** is the Hessian matrix of the free energy ($F \equiv Nf$)

$$A_{ij} = \frac{\partial^2 F}{\partial m_i \partial m_j} = a_i \delta_{ij} - \beta J_{ij} - \frac{2\gamma\beta^2 J^2}{N} m_i m_j \tag{2.3}$$

$$a_i = \frac{1}{1 - m_i^2} + \gamma\beta^2 J^2 (1-q). \tag{2.4}$$

The integrals over λ, u and x_i have been introduced to represent the restrictions $q = (1/N)\sum_i m_i^2$, $f = (1/N)\sum_i f_\gamma(m_i)$ and $G_i = 0$, respectively.

As pointed out by BM1, since $N_s(f) \sim \exp(\alpha N)$ is expected, we have strictly to average $\ln N_s(f)$ over the realisations of $\{J_{ij}\}$. But BM1 have also argued on the TAP equations that the direct average $\langle N_s(f)\rangle_J$ is enough to evaluate the total number of metastable states, $N_s \equiv \int \mathrm{d}f\, N_s(f)$, which is of present interest. Expecting this is also the case for models with γ less than unity, we examine here only $\langle N_s \rangle_J$.

After some manipulation briefly described in the Appendix we end up with the following stationary-point problem; $\langle N_s \rangle_J$ is given by

$$\langle N_s \rangle_J = \max_{[\lambda,q,B,\Delta,\sigma]} \{\exp(N\phi)\} \tag{2.5}$$

$$\phi = -\lambda q + \tfrac{1}{2}t^2(B^2 - \Delta^2 - 2\sigma B - 2\sigma^2) - \gamma l(\Delta + B - \sigma)(1-q) + \ln \mathscr{I} \tag{2.6}$$

$$\mathscr{I} = \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{(2\pi)^{1/2}} \frac{\{1 + lB(1-m^2)\}^{3/2}}{\{1 + l(B+2\sigma)(1-m^2)\}^{1/2}} \exp\left[-\frac{1}{2}\left(z - \frac{t\Delta}{q^{1/2}}m\right)^2 + \lambda m^2\right]. \tag{2.7}$$

Here $t = T/T_c(\gamma)$, $l = J/T_c(\gamma)$ and $m = \tanh(\beta J q^{1/2} z)$, $T_c(\gamma)$ being the transition temperature, which is given by equation (2.10*a*) below. The stationary equations for the parameters $\lambda, q, B, \sigma, \Delta$ are given by

$$q = \langle m^2 \rangle_z \tag{2.8a}$$

$$\lambda = \gamma l(\Delta + B - \sigma) - \frac{1}{2q}\left[1 - \left\langle\left(z - \frac{t\Delta}{q^{1/2}}m\right)^2\right\rangle_z\right] \tag{2.8b}$$

$$\Delta = -\frac{1}{2t^2}\left(\gamma l(1-q) - \frac{t}{tq^{1/2}}\langle zm\rangle_z\right) \tag{2.8c}$$

$$B = \frac{1}{t^2}\left(\gamma l(1-q) - \left\langle \frac{l(1-m^2)}{1 + lB(1-m^2)}\right\rangle_z\right) \tag{2.8d}$$

$$\sigma = \frac{1}{2t^2}\left(\left\langle \frac{l(1-m^2)}{1 + lB(1-m^2)}\right\rangle_z - \left\langle \frac{l(1-m^2)}{1 + l(B+2\sigma)(1-m^2)}\right\rangle_z\right) \tag{2.8e}$$

where $\langle \ldots \rangle_z$ represents the average over z with the weight given by the integrand of equation (2.7). We note that by the present formulation the linear, uniform susceptibility, χ_0, is given by

$$J\chi_0 = \lim_{h_i \to 0} J\left\langle \frac{\partial m_i}{\partial h_i}\right\rangle_J = \frac{1}{t}\left\langle \frac{l(1-m^2)}{1 - l\Delta(1-m^2)}\right\rangle_z. \tag{2.9}$$

Before going into the search of solutions of equations (2.8), it is worth mentioning here the stability of the paramagnetic point, $\forall m_i = 0$. Since the lowest eigenvalue of the Hessian matrix $\mathbf{A}$ at this point is given by $1 - 2\beta J + \gamma\beta^2 J^2$, it is unstable in the temperature range shown in figure 1. Its upper boundary is nothing but the spin glass transition temperature $T_c(\gamma)$ given by

$$T_c(\gamma)/J = (1 + \sqrt{1-\gamma}) \equiv 1/l. \tag{2.10a}$$

We denote, on the other hand, the lower boundary by $T_c^\star(\gamma)$, which is given by

$$T_c^\star(\gamma)/J = (1 - \sqrt{1-\gamma}) \equiv 1/l^\star. \tag{2.10b}$$

The apparent recovery of the paramagnetic stability at lower temperatures is due to the Onsager term in equation (1.2). It is known for $\gamma = 1$, however, that the paramagnetic point is in the region where the validity condition of the TAP equations themselves is violated [3, 11]. The expected free energy on the axis connecting this point and one of the spin glass solutions is shown schematically in inset (*a*) of figure 1 [5], where the broken line represents a branch in the invalid region of the equations. On the other hand, the corresponding plot for the NMF equations is shown in inset (*b*). We may deduce that the critical line $T_c^\star(\gamma)$ separates these two characteristic features.

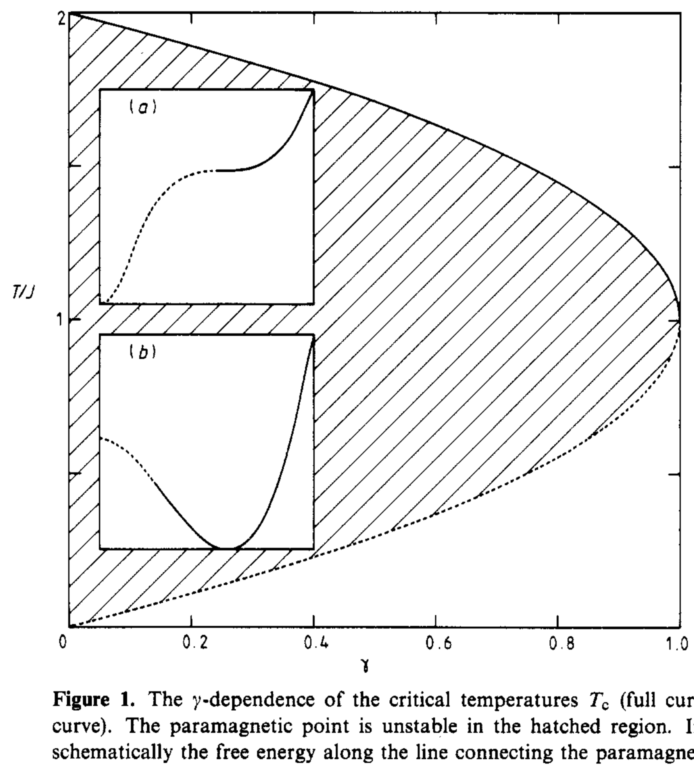


Figure 1. The γ-dependence of the critical temperatures T_c (full curve) and $T_c^\star$ (broken curve). The paramagnetic point is unstable in the hatched region. In the insets we plot schematically the free energy along the line connecting the paramagnetic point and one of the metastable points (spin glass states) for $T < T_c^\star$ (*a*) and for $T_c^\star < T < T_c$ (*b*), where the part represented by the broken curve is in the invalid region.

For the TAP equations ($\gamma = 1$) BM1 looked for solutions by setting $\sigma = 0$ from the start, and found three spin glass solutions ($q > 0$) below T_c: the SK solution for which only q is non-zero, the Sommers solution [10] with $B = -\Delta < 0$, and the one with

$B = 0$ but with non-vanishing Δ and λ. Only the last solution, which we call the $\mathrm{BM_I}$ solution, has non-zero ϕ. Its temperature dependence is shown by the broken line in figure 2. In the present investigation we find three other solutions. One is with $\sigma = 0$ and it has a little larger ϕ than the $\mathrm{BM_I}$. We call it the $\mathrm{BM_{II}}$ solution. The second one, which we call the modified SK (MSK) solution, is with $\sigma > 0$ and has much smaller but non-vanishing ϕ. The last one is with $\sigma < 0$ and non-vanishing ϕ. We name it the general (GE) solution.

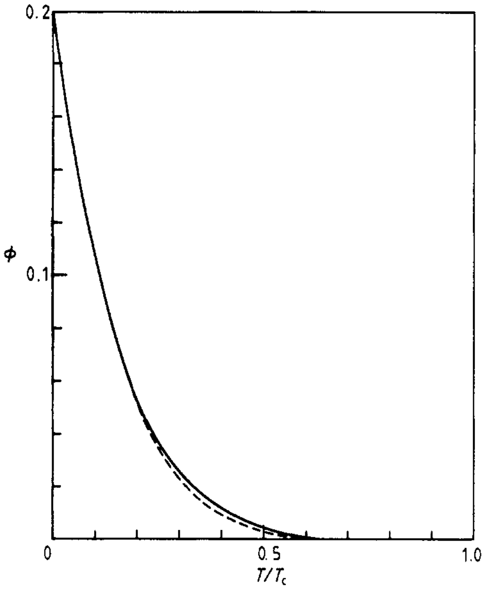


Figure 2. Temperature dependence of $\phi = (1/N)\ln\langle N_s\rangle_J$, N_s being the number of metastable states. The full and broken curves represent ϕ of the GE solution for the NMF equations and that of the $\mathrm{BM_I}$ solution for the TAP equations, respectively.

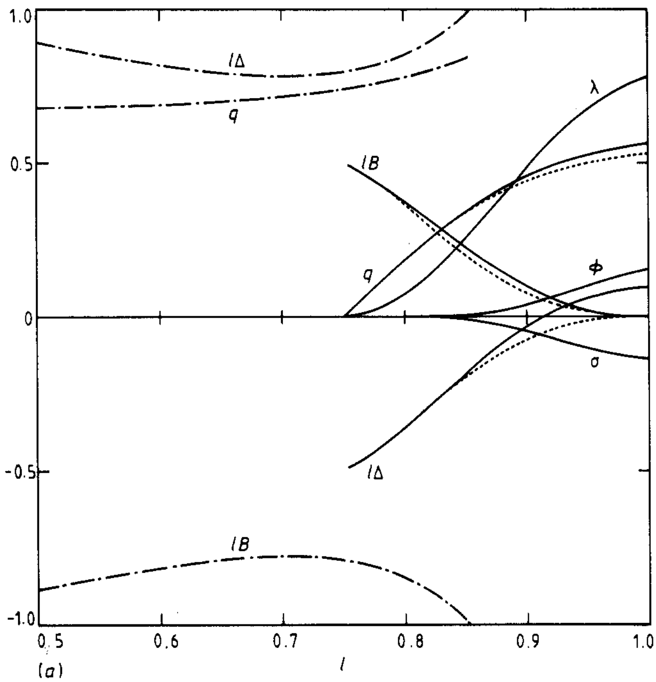
For models with γ less than unity we find the solutions that are smoothly connected to each of the above six solutions in the limit $\gamma \to 1$. An example of such a trace at a fixed reduced temperature $t \equiv T/T_c(\gamma) = 0.5$ is shown in figure 3, where the various parameters are plotted against l which is related to γ through equation (2.10a).

The inspection of such traces reveals the following aspects of the six solutions.

(1) *SK solution* ($\lambda = \sigma = \phi = 0$). As l decreases from unity, $B = -\Delta > 0$ starts to grow, while q decreases (figure 3(a)). An interesting observation here is that this solution disappears at a critical value of l with vanishing q. We can ascertain analytically from equations (2.8) that this critical point just coincides with l^* of equation (2.10b). This means that the SK solution exists only when the paramagnetic point is apparently stable (inset (a) in figure 1). Its growth is triggered by the instability of that point at T_c^*. Note that for the SK model T_c^* coincides with T_c.

(2) *MSK solution.* As seen in figure 3(a), this solution is quite close to the SK solution, though it associates with non-vanishing λ, σ and ϕ. It also vanishes at l^*, where it merges to the SK solution (q, λ, σ and ϕ also vanish, while B and Δ remain finite).

(3) *Sommers solution* ($\lambda = \sigma = \phi = 0$, $B = -\Delta < 0$). For the NMF equations this solution coincides exactly with the replica symmetric solution of the BSY model [8], and B is related to the susceptibility $\tilde{\chi} \equiv \beta^{-1}\chi_0$ of BSY by $B = -2\beta^2J^2\tilde{\chi}$. As Sommers



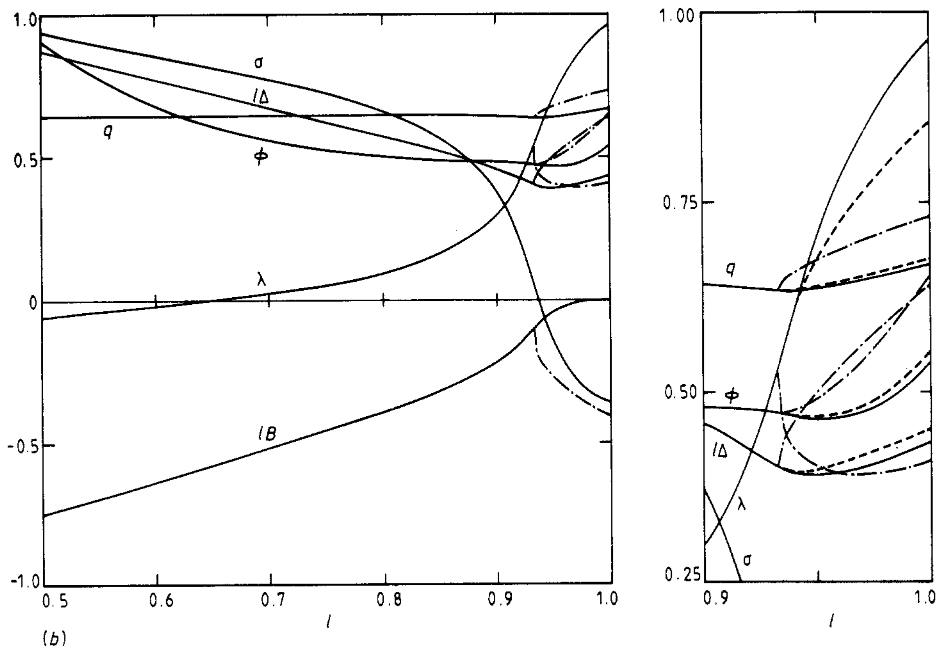


Figure 3. (a) The l-dependence of the parameters $\lambda, q, l\Delta, lB, \sigma$ and ϕ at $t = T/T_{\mathrm{c}}(l)$ for the MSK (full), the SK (broken) and the Sommers solutions (chain). The unit of abscissa is for q, $l\Delta$ and lB, while λ, σ and ϕ are arbitrarily scaled. (b) As (a) but for the GE (full), the $\mathrm{BM_I}$ (broken) and the $\mathrm{BM_{II}}$ solutions (chain). The enlarged plots of the region $0.9 \leq l \leq 1$ are presented in the right-hand figure.

[10] and BSY pointed out for the $l = 1$ and $l = 0.5$ cases, respectively, this solution has another critical point at $T = T_S^\star(l)$. In terms of the present formulation, lB in the region $T > T_S^\star(l)$ satisfies $|lB| < 1$, the condition that the integrals by steepest descents to derive equation (2.5) are convergent. This condition is violated in the low-temperature region. Therefore we plot in figure 3a the Sommers solution only in the high-temperature region, and present the l-dependence of $T_S^\star/T_c$ in figure 4.

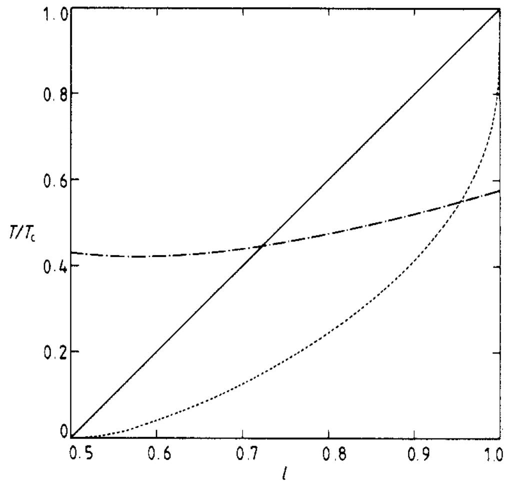


Figure 4. The critical lines $t_c^\star = T_c^\star/T_c$ (full), $t_\sigma^\star = T_\sigma^\star/T_c$ (broken) and $t_S^\star = T_S^\star/T_c$ (chain) in the l–t plane.

(4) *BM$_{\rm I}$ and BM$_{\rm II}$ solutions* ($\sigma = 0$). These two solutions are a pair of roots of equations (2.8*c*) and (2.8*d*) in the parameter space of B and Δ. As seen in figure 3(*b*), they merge together at $l = l_\sigma^\star$ and disappear in the range $l < l_\sigma^\star$. The corresponding critical line $t_\sigma^\star(l)$, the inverse of $l_\sigma^\star(t)$, is also shown in figure 4.

(5) *GE solution.* This solution is a most general one in the sense that all five parameters as well as the ϕ associated with it are non-vanishing. As we see in figure 3(*b*), σ crosses zero at $l = l_\sigma^\star$, where it touches with the BM$_{\rm I}$ and BM$_{\rm II}$ solutions (see the enlarged figure).

The largest value of ϕ is taken by the GE solution for $l < l_\sigma^\star$ and by the BM$_{\rm II}$ solution for $l > l_\sigma^\star$. At any reduced temperature ϕ of the NMF model is largest and is plotted by the solid curve in figure 2. The absolute magnitudes of the differences in ϕ at a fixed t are rather small within the scale of figure 2, so that the overall t-dependence of ϕ looks similar to all the mean-field models of present interest. However, its relative differences are already significant at $t = 0.5$ as seen in figure 3(*b*), and they grow further as t approaches unity, as will be discussed just below.

For further comparison of the SK and NMF models we present some asymptotic expressions for the GE solution of the NMF model. Near T_c it is solved as

$$q = \epsilon + 2\epsilon^2 - 14\epsilon^3 + \tfrac{2078}{9}\epsilon^4 + \dots \tag{2.11a}$$

$$\Delta = 1 + \epsilon - \tfrac{1}{3}\epsilon^2 + 13\epsilon^3 + \dots \tag{2.11b}$$

$$B = -1 - \epsilon + \tfrac{5}{3}\epsilon^2 + \tfrac{137}{3}\epsilon^3 + \dots \tag{2.11c}$$

$$\sigma = \tfrac{4}{3}\epsilon^2 + \tfrac{176}{3}\epsilon^3 + \ldots \tag{2.11d}$$

$$\lambda = -\tfrac{8}{3}\epsilon^3 + \ldots \tag{2.11e}$$

where $\epsilon = 1 - t$. It is noted that the equality $B = -\Delta$ breaks down from the order of ϵ^2, while q and $B + \Delta - \sigma$ start to deviate from the corresponding quantities of the Sommers solution from the order of ϵ^4 and ϵ^5, respectively. Using the above solutions we obtain

$$\phi_{\rm NMF} = \tfrac{128}{21}\epsilon^6 + \ldots . \tag{2.12}$$

The leading order contribution to $\phi_{\rm NMF}$ is proportional to ϵ^6, the same as the BM1 result for the TAP equations. But the coefficient of the latter is about 50 times smaller than the above result.

Near $T = 0$, $q \simeq 1 - \alpha t^2$, and so the Onsager term vanishes at $T = 0$, both in the GE solution of the NMF equations and in the BM1 solution of the TAP equations. For the NMF equations we obtain $\Delta \cong 0.5060/t$, while B, σ and λ do not diverge. For the TAP equations, on the other hand, $\Delta \cong \lambda \cong 0.5060/t$. Both solutions yield the same result $\phi(T = 0) \cong 0.1993$, as expected [12].

We conclude that the Onsager term does not play any crucial role in the actual existence of many metastable states in the spin glass phase, but only in determining their number. It has to be noted again here that, as pointed out in the Appendix, this conclusion is based on the analysis of the replica symmetric solution of $\langle N_{\rm s}\rangle_J$.

3. Marginal stability of the NMF spin glass phase

From the schematic plots of the free energy shown in insets (*a*) and (*b*) of figure 1 the marginal stability of the spin glass phase is apparent for the SK model but not for the NMF model. Therefore we examine numerically the spin glass susceptibility $\chi_{\rm SG}$ to check the stability of the latter by means of the method of Bray and Moore (BM2) [4]. It is given by

$$\chi_{\rm SG} = \frac{\beta^2}{N}\,{\rm Tr}\,\mathbf{A}^{-2} = \beta^2 \int {\rm d}\lambda\, \frac{\rho(\lambda)}{\lambda^2} \tag{3.1}$$

where $\mathbf{A}$ is the Hessian matrix defined by equation (2.3) and $\rho(\lambda)$ is its spectrum of eigenvalues. We have solved the NMF equations (equation (1.1) with $\gamma = 0$) numerically. A set of equations $(\partial m_i/\partial t) = -\{m_i - \tanh(\beta \sum_j J_{ij} m_j)\}/\tau$ is solved to find a stationary solution starting from that at the preceding temperature, where τ is the relaxation time properly chosen [7]. Only the cooling process has been examined in the present work. The solution thus obtained is expected to be one of metastable states with lowest free energies. We have carried out one cooling process for each realisation of $\{J_{ij}\}$. The numbers of samples examined are 500, 162, 65 and 42 for $N = 50$, 100, 200 and 400, respectively.

We show in figure 5 the results of $\langle \lambda_{\rm min}\rangle_J$, the lowest eigenvalue of the Hessian matrix, and $\langle \chi_{\rm SG}^{-1}\rangle_J$. For a fixed N both quantities tend to increase as temperature decreases. But they vanish in the thermodynamic limit at each temperature. Their power law behaviours in N, i.e., $\langle \lambda_{\rm min}\rangle_J \simeq N^{-2/3}$ and $\langle \chi_{\rm SG}^{-1}\rangle_J \simeq N^{-1/3}$, are identical to the corresponding BM2 results for the TAP equations. They indicate $\rho(\lambda) \propto \lambda^{1/2}$ for $\lambda \simeq 0$.

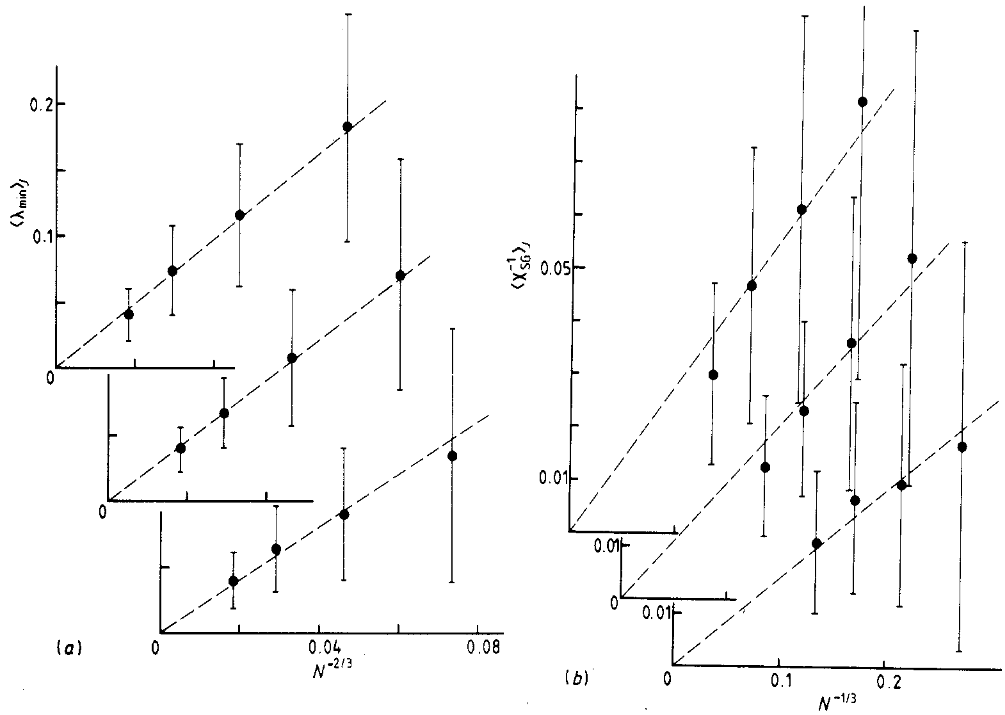


Figure 5. Sample-size dependence of $\langle\lambda_{min}\rangle_J$ (*a*) and $\langle\chi_{SG}^{-1}\rangle_J$ (*b*). The results at $t = 0.5$, 0.7 and 0.9 are presented from the top to the bottom. The broken lines are simply guides for the eye.

The above results confirm that the spin glass phase of the NMF model is also marginally stable, and that the Onsager term is not indispensable for yielding the marginal stability. Combined with the expected free energy plot in inset (*b*) of figure 1, it is suggested that the direction of the marginal stability is almost perpendicular to the direction towards the paramagnetic point.

In order to get further insights into the spin glass phase of the NMF model, we examine the projection of the states to the maximum eigenmode, $\{\langle\Lambda|i\rangle\}$, of the J_{ij} matrix. In figure 6 the data of $q^{-1}[(1/N)\sum_i\langle\Lambda|i\rangle m_i]^2$ are presented. We can read from the figure that in the limit $N \to \infty$ the macroscopic condensation to the Λ-mode occurs just at T_c, but the mode-mixing takes place immediately below T_c. Although our present data of $q^{-1}[(1/N)\sum_i\langle\Lambda|i\rangle m_i]^2$ remain finite at lower temperatures, which may be partly due to our method of simulation, i.e., gradual cooling, and partly due to small sizes of our samples investigated, they may support Sompolinsky's picture [13] that the dominant condensation to the Λ-mode breaks down immediately below T_c. It is of interest to examine whether this type of mode-mixing is related to the idea of marginal stability being due to the continuous bifurcation of spin glass states [14, 15].

4. Conclusions

We have shown by explicit evaluation of the number of metastable states and the spin glass susceptibility that the NMF model shares the novel spin glass properties such as

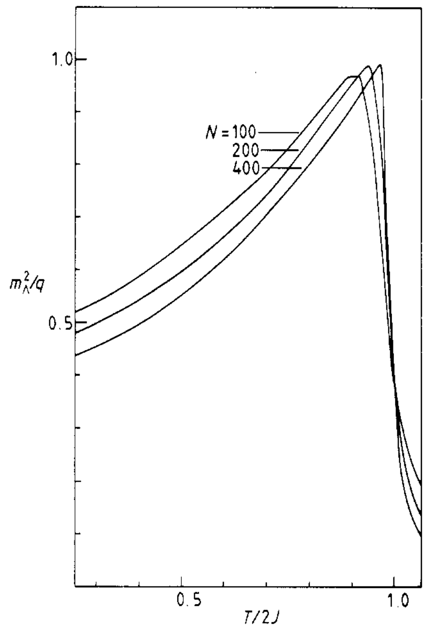


Figure 6. The plots of $q^{-1}[(1/N)\sum_i\langle\Lambda|i\rangle m_i]^2$, where $\{\langle\Lambda|i\rangle\}$ denotes the maximum eigenmode of the J_{ij} matrix.

the rugged free energy structure and the marginal stability first found by the mean-field theory based on the SK model. It is expected that they are common to the class of mean-field models introduced by the present work. Since these models coincide at $T=0$ (because $q \simeq 1-\alpha t^2$), it is also plausible that the metastable states exhibit the ultrametric organisation as those of the SK model [16]. We believe that these spin glass properties are intrinsic ones common to various randomly frustrated systems and that they are not necessarily associated with the replica-symmetry-breaking concept.

Acknowledgments

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Appendix

We make here brief comments on the derivation of equations (2.5)–(2.7) from equation (2.1). By a simple translation of each integral variable J_{ij} in the expression of $\langle N_s(f)\rangle_J$ the average over $\{J_{ij}\}$ can be confined into $\langle \det \tilde{\mathbf{A}}\rangle_J$, where $\tilde{A}_{ij} = A_{ij} + (1/N)\beta^2 J^2(x_i m_j + x_j m_i)$. BM1 neglected this additional term as well as the last term in equation (2.3) simply because they are of order N^{-1}. This argument does not hold true strictly as already pointed out by Plefka [11]. Actually, keeping these terms, and introducing the replica method of BM1, we obtain

$$
\begin{aligned}
\langle N_s(f)\rangle_J = \max_{[q,\lambda,u,P,Q,R,S,V]} \int_{-1}^{1} \mathrm{d}m_i \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \prod_i \left(\frac{\mathrm{d}x_i}{2\pi \mathrm{i}}\right) \\
\times \exp\Bigg[N\{-\lambda q - uf - \tfrac{1}{2}V^2 - 2PQ - 2\gamma Q^2 + 2R^2 - \tfrac{3}{2}S^2\} \\
+ \sum_i \left(\frac{1}{2}\beta^2 J^2 q - \frac{\beta^2 J^2 Q^2}{\tilde{a}_i + 2\beta J S}\right) x_i^2 + \sum_i \left(g_\gamma(m_i) + \beta J V m_i - \frac{2\beta^2 J^2 P Q m_i}{\tilde{a}_i + 2\beta J S}\right) x_i \\
+ \sum_i \left(\lambda m_i^2 - \frac{2\beta^2 J^2 P^2 m_i^2}{\tilde{a}_i + 2\beta J S}\right) + u \sum_i f(m_i)\Bigg] \prod_i \tilde{a}_i \left(+ \frac{\tilde{a}_i}{\tilde{a}_i + 2\beta J S}\right)^{1/2} \qquad (\mathrm{A}.1)
\end{aligned}
$$

where $\tilde{a}_i = a_i - 2\beta JR + \beta JS$, and S corresponds to the parameter $T_{\alpha\beta}$ in BM1, i.e., we put $T_{\alpha\beta} = S$ for all the pairs of $\alpha\beta$. Also to introduce R, V, P and Q similar replica symmetry has been assumed. The parameters P and Q above come out by applying the Hubburd–Stratonovich identity to the terms neglected by BM1. As we see from this equation, there seems no simple reasoning that these contributions vanish *a priori.*

Equation (A.1) represents, after the integrations over x_i, a stationary-point problem in the space of the eight parameters, as indicated. Since the search for solutions in the whole parameter space is a formidable task, we reduce it by putting $P = Q = 0$, which certainly solves the stationary equations. The stability analysis of solutions is even harder since it requires full information on the Hessian matrix in the infinite-dimensional replica space, i.e., we have to perform an analysis similar to that of de Almeida and Thouless [17] in such a space. In the present work we do not go into such a stability analysis, but only look for stationary solutions with $P = Q = 0$.

Furthermore, in order to obtain the total number of metastable states $\langle N_s\rangle_J$ we put $u = 0$. Analogously to BM1, we then change the parameters so that $\gamma\beta^2J^2(1-q)-2\beta JR = lB$, $\gamma\beta^2J^2(1-q) + \beta JV = -l\Delta$, and $\beta JS = l\sigma$, and obtain equations (2.5)–(2.7) in the text.

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